IBDP HL Mathematics: Analysis and Approaches

Extended Essay

Research Title: *Optimising Happiness Levels of Students: An Investigative Study of the Methods of Multivariable Optimisation*

Research Question: How can we use functions and calculus to optimise the satisfaction of IB students, dependent on studying and playing?

Word Count: 3999

To Apala for introducing me to the Work-Leisure trade-off,

To my father for his continued support in all my endeavours,

And to the people who are never happy no matter what they do.

CONTENTS

Introduction

Ever since its conception in the 17th century, *multivariable calculus* has permeated every aspect of our lives. From the way we study the natural world to the way we construct our artificial one, modelling change and obtaining optimality have become fundamental to our social progress. With the development of these mathematical tools, I felt empowered to study one of humanity's most profound questions, "How can we be the happiest?".

This question is of personal significance to me as I have always had trouble figuring out which combination of academic work to leisurely activities would leave me most satisfied. As I spoke to others, I realised this was an issue many of us experienced, especially considering the emphasis our Asian backgrounds placed on education, and so this became the focus of my investigation.

As I conducted research in this area, I was not surprised to find that many other academics had also grappled with similar problems. In fact, I found that this issue was so widely studied in Economics, that the term happiness levels was coined as *utility*. Not only that, but in an attempt to optimise these happiness levels, Economists had also conceptualised *utility functions.* Though these *functions* are typically used in the context of material good *consumption*, I felt that reapplying them to other use cases such as time spent studying or playing would only lead to more meaningful outcomes, where perhaps a mathematical response could provide those unsatisfied with some direction.

Hence, I arrived at my research question, "How can we use functions and calculus to optimise the satisfaction of IB students, dependent on studying and playing?". To answer this question, I collected *sample data* on the happiness levels of IB students in my grade. Borrowing *utility functions* from the human sciences, I set up a *mathematical model* to represent their happiness levels and optimise them. By reappropriating the *Cobb Douglas function*, I not only demonstrated its applicability to new situations but also, its ability to provide new solutions. However, most importantly, by illustrating the variety of ways to solve this problem using *derivatives, contour lines* and the *Lagrange multiplier*, I attempted to analyse and evaluate the current methods of *mathematical optimisation*, thus ascertaining the optimal ratio of study to play and the most effective method of solution.

Modelling Happiness Levels Dependent on One Variable

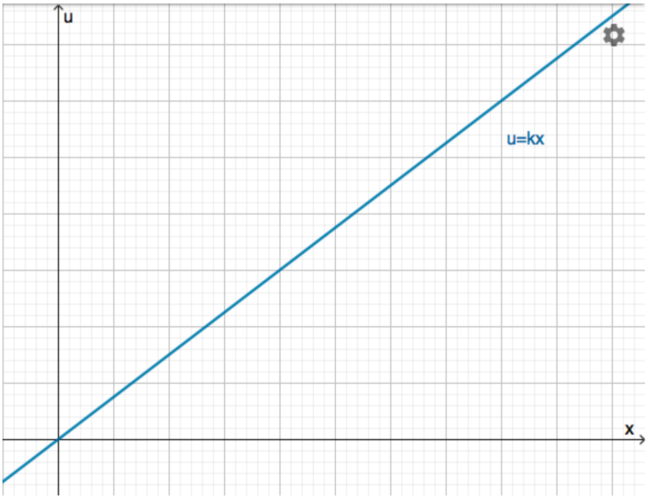
Linear Functions

To begin investigating this problem, I had to quantify happiness. To do so I borrowed the concept of *utility: a numeric measure of a person's happiness or a means of describing a person's preferences.*[1] I also had to establish my *variables*. If happiness levels, or more formally utility, was my *dependent variable*, I would need an *independent variable*. For simplicity, let's assume that the number of hours we spend with our friends affects our utility. It would be intuitive to assume as one increases the other would increase, allowing us to represent this relationship graphically.

Utility *(u)*

Number of hours spent with friends *(x)*

Key for Graph 1.1



Graph 1.1: u plotted against x where u is directly proportional to x i.e. u(x)=kx.

[1]"Utility." *Intermediate Microeconomics: A Modern Approach*, edited by Jack Repcheck, Eighth Edition, W.W Norton & Company, 2010, p. 54.

The *linear function* may have a *y-intercept* depending on the initial nature of its utility. To accurately sketch this function, more data would have to be collected to determine its behaviour at the *origin*.

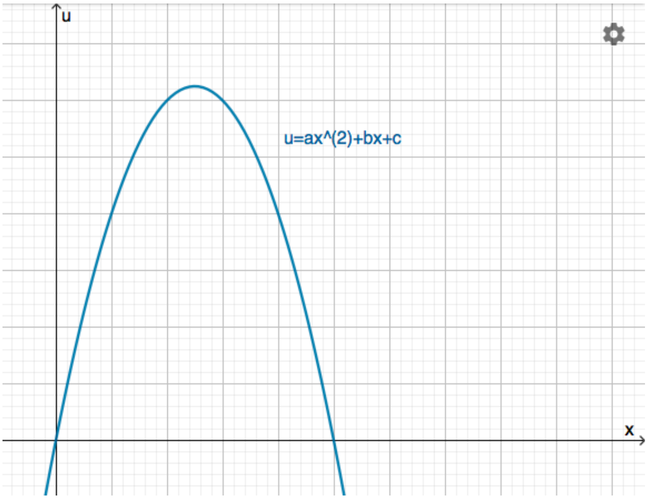
<u>Quadratic Functions</u>

However, we know that our utility does not increase without end as we spend time with our friends. It is likely that after a certain period of time, we would get annoyed or bored. Instead, it is more likely that we would reach a *satiation point*, after which our utility would decrease as we continued to spend time with our friends. Thus, we would represent the relationship as shown in Graph 1.2.

Utility *(u)*

Number of hours spent with friends *(x)*

Key for Graph 1.2



Graph 1.2: u plotted against x where u and x share a quadratic relationship i.e. $u(x)=ax^2+bx+c$.

The *quadratic function* may continue to decrease like a normal *upside-down parabola* or it may continue *asymptotic* to the *x-axis,* depending on the nature of its utility.

Modelling Happiness Levels Dependent on Two Variables

Introduction to the Cobb Douglas Function

However, single variable relationships are insufficient to model our utility. This is because our utility is dependent on many variables. Instead, we will have to use a multivariable model.

For this investigation, I felt it would be most appropriate to modify and utilise the Cobb Douglas function. This is because the function considers the *preferences* of two variables at the same time. In this case, *hours spent studying* and *playing*.

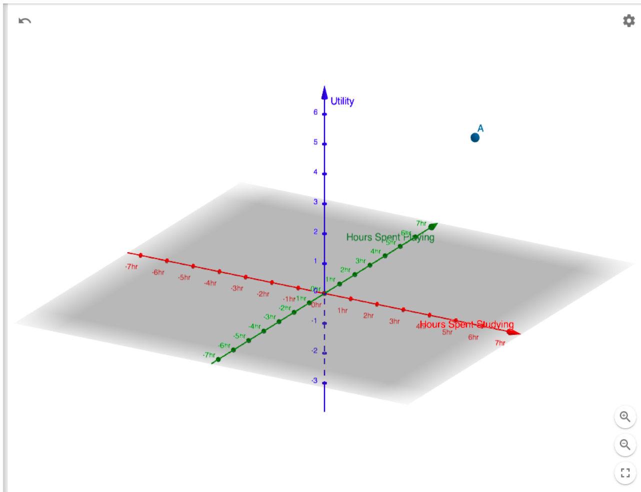
But before I explain the other, more nuanced, reasons for why the Cobb Douglas function would be appropriate to model the happiness levels of IB students, dependent on studying and playing, I will first, give a bit of background information on utility functions.

Introduction to Utility Functions

A utility function is a way of assigning a number to every possible consumption bundle such that more-preferred bundles get assigned larger numbers than less-preferred bundles.[2] A consumption bundle is a certain combination of the two variables being consumed at certain quantities. The larger the number assigned, the more preferred and the more utility derived from the consumption bundle.

In this case, consumption bundles refer to the combinations of hours spent on studying and playing. Mathematically, this means (x, y, z) in the $x - y - z$ *plane* will represent (hours spent studying, hours spent playing, utility derived).

[2]"Preferences." *Intermediate Microeconomics: A Modern Approach*, edited by Jack Repcheck, Eighth Edition, W.W Norton & Company, 2010, p. 33-37.



Graph 2.1: Point A (4,3,5) represents 4 hours spent studying and 3 hours spent playing at the utility of 5.

<u>Assumptions of the Cobb Douglas Function</u>

For utility to be modelled by the Cobb Douglas function, it must agree with the following assumptions. Note, these assumptions exist mainly to make the problem mathematically tractable.

- The concerned utility must be *complete, reflexive* and *transitive*.[3] This means that the utility derived from hours spent studying and playing can be ranked and follows a consistent order
    - In other words, some combinations of hours spent studying and playing are preferred over others.[4]

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[3] "Preferences." *Intermediate Microeconomics: A Modern Approach*, edited by Jack Repcheck, Eighth Edition, W.W Norton & Company, 2010, p. 35.

[4] "Preferences." *Intermediate Microeconomics: A Modern Approach*, edited by Jack Repcheck, Eighth Edition, W.W Norton & Company, 2010, p. 35.

- And if combination A is preferred over combination B and combination B is preferred over combination C, then combination A is preferred over combination C.[5]

These assumptions apply to the utility derived from hours spent studying and playing and thus, the Cobb Douglas function can be used to model this utility.

<u>Properties of the Cobb Douglas Function</u>

However, the Cobb Douglas function also has a specific form $u(x, y) = cx^a y^b$ which causes it to have certain properties appropriate for modelling this kind of utility. Thus, before I begin modelling this utility, I will explain the properties of the utility derived from hours spent studying and playing and how they agree with the properties of the function.

- Earlier, we established that our utility depends on two or more variables i.e. hours spent studying and playing. Mathematically, this means $u(x, y)$.
- Furthermore, we know the more we consume i.e. the more hours spent on studying and playing, our utility increases. As our consumption of x or y increases, u increases. Mathematically, this means $\frac{\partial u}{\partial x} > 0$ and $\frac{\partial u}{\partial y} > 0$.[6]
- We also know that the more we consume i.e. the more hours spent on studying and playing, our utility increases but slower and slower. As our consumption of

[5] "Preferences." *Intermediate Microeconomics: A Modern Approach*, edited by Jack Repcheck, Eighth Edition, W.W Norton & Company, 2010, p. 35.

[6] Tragakes, Ellie. "Chapter 2: Competitive Markets: Demand and Supply." Economics for the IB Diploma, Second Edition, Cambridge University Press, 2012, p. 22.

x or y increases, u increases at a decreasing *rate*. Mathematically, this means $\frac{\partial^2 u}{\partial x^2} < 0$ and $\frac{\partial^2 u}{\partial y^2} < 0$.[7]

- Lastly, we know that we consume in a balanced way. We prefer consuming a bit of both as opposed to a lot of one and a little of another i.e. we prefer spending a few hours studying and playing as opposed to almost all our hours studying and a few playing or vice versa. As our consumption of x decreases, our consumption of y grows *hyperbolically* to compensate for the loss in balanced consumption. Mathematically, this means $y^b \propto \frac{1}{x^a} \Rightarrow y^b = \frac{k}{x^a}$which also means, the *contour lines* of $u(x, y)$ are hyperbolic $\Rightarrow u(x, y) = cx^a y^b$. Note, k and c are *constants* which behave as *scaling factors* and do not change this hyperbolic property.[8]

For $u(x, y) = cx^a y^b$, all of the above properties hold. Thus, the Cobb Douglas function can be used to model this utility.

[7]Tragakes, Ellie. "Chapter 2: Competitive Markets: Demand and Supply." Economics for the IB Diploma, Second Edition, Cambridge University Press, 2012, p. 22.

[8]"Preferences." *Intermediate Microeconomics: A Modern Approach*, edited by Jack Repcheck, Eighth Edition, W.W Norton & Company, 2010, p. 46.

However, before $u(x, y) = cx^a y^b$ is used for modelling, it must be modified to $u(x, y) = cx^a y^{1-a}$. This is because we consume hours spent studying and playing in a day of 24 hours. As we spend more hours studying each day, we must *proportionally* spend fewer hours playing. Mathematically, this means $b = 1 - a \Rightarrow u(x, y) = cx^a y^{1-a}$.

To model the utility derived from hours spent studying and playing, we have to first apply *linear law* to $u(x, y) = cx^a y^{1-a}$so that the data collected can be used to find the unknown variables.[9]

$$u(x, y) = cx^a y^{1-a}$$

Taking the natural logarithm on both sides give us

$$lnu = lncx^a y^{1-a}$$

$$lnu = lnc + alnx\ + lny - \ alny$$

Factoring out a gives us

$$lnu - lny = a(lnx\ - lny) + lnc$$

$$Let\ Y = lnu - lny, M = a, X = lnx - lny, B = lnc$$

$$Y = MX + B$$

Raw Data

The *raw data* collected shows how satisfied NPSi Singapore IB year 1 and 2 students are when studying and playing for a certain number of hours over the course of a day,

[9]Riveros, John. "A Brief Example to Model the Cobb-Douglas Utility Function Using Stata." *MSR Economic Perspective*, MSR Economic Perspective, 19 Dec. 2019, blog.ms-researchhub.com/2019/12/19/a-brief-example-to-model-the-cobb-douglas-utility-function-using-stata. Accessed 30 Sep. 2020.

after school and before sleeping. This raw data was collected by taking a *voluntary survey*.

x Hours spent studying after school before sleeping i.e. any work related to but not limited to HLs, SLs, EE or TOK (calibrated on a scale of 0-24)

y Hours spent playing after school before sleeping i.e. any work related to but not limited to social media, television, hobbies or CAS (calibrated on a scale of 0-24)

u% Satisfaction with hours spent studying and playing after school and before sleeping (in percentage)

u Satisfaction with hours spent studying and playing after school and before sleeping (calibrated on a scale of 0-24)

Key for Table 1.1

Candidate Responses	*x*	*y*	*u%*	*u*
A	3.00	11.0	80.0	19.2
B	4.00	2.00	60.0	14.4
C	4.00	3.00	35.0	8.40
D	6.00	4.00	65.0	15.6

Table 1.1: Raw data on hours spent studying, playing and satisfaction → completed table in the appendix.

Processed Data

This *processed data* was obtained by taking the natural logarithm of the raw data and using this *formula Let* $Y = lnu - lny, M = a, X = lnx - lny, B = lnc$.

x Hours spent studying after school before sleeping i.e. any work related to but not limited to HLs, SLs, EE or TOK *y* Hours spent playing after school before sleeping i.e. any work related to but not limited to social media, television, hobbies or CAS

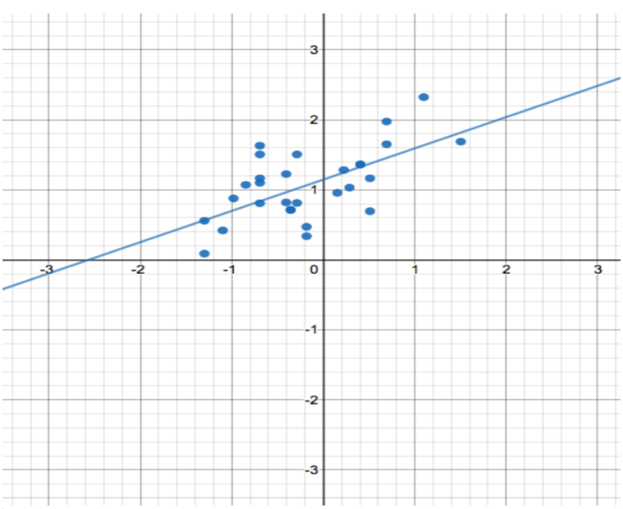
u Satisfaction with hours spent studying and playing after school and before sleeping

Key for Table 1.2

			Coordinates	
lnx	*lny*	*lnu*	*X = lnx-lny*	*Y = lnu-lny*
1.10	2.40	2.95	-1.30	0.557
1.39	0.693	2.67	0.693	1.97
1.39	1.10	2.13	0.288	1.03
1.79	1.39	2.75	0.405	1.36

Table 1.2: Processed data of hours spent studying, playing and satisfaction → completed table in the appendix.

This *line of best fit* was obtained by using the processed data.



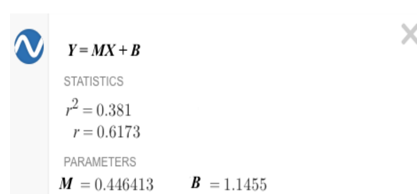


Figure 1.1: Legend of Graph 3.1 where M = a = 0.446 and B = lnc = 1.15.

Graph 3.1: Y = lnu-lny plotted against X = lnx-lny with line of best fit Y = MX + B where M = a and B = lnc.

Resultant Model

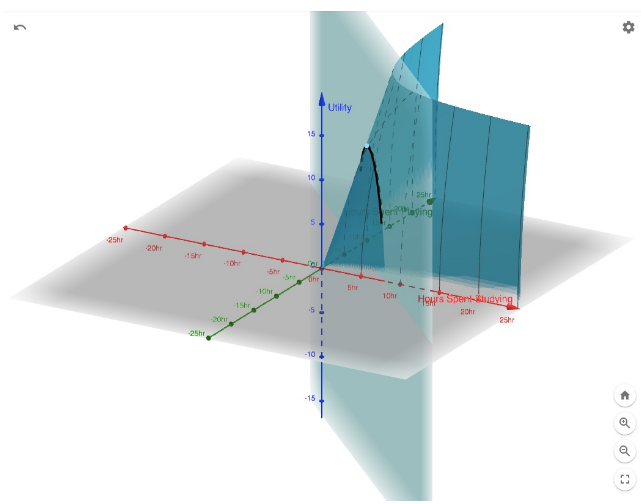
From Graph 3.1, we have found that $a = 0.446 \therefore b = 1 - 0.446 = 0.554$ .We have also found that $c = e^{1.15} = 3.16$. Hence, $u(x, y) = 3.16x^{0.446}y^{0.554}$ is the Cobb Douglas function which models the satisfaction of IB students, dependent on studying and playing.

However, this utility is subject to *constraints*. Considering that a day has 24 hours and that school occupies 8 and sleep occupies another 8, hours spent studying and playing are constrained to the hours available after school and before sleeping. Mathematically, this means $x + y \leq 8$. However, earlier we established as we consume more, our utility increases. Thus, $u(x, y) = 3.16x^{0.446}y^{0.554}$ must be optimised with respect to the constraint $x + y = 8$.

Note, data of *three significant figures* are used, as a result, *negligible errors* should be ignored.

Differentiation

If we look at this problem graphically, it is not too difficult to solve using *differentiation*. Graph 4.1 shows our utility function $u(x, y) = 3.16x^{0.446}y^{0.554}$ in dark blue with our constraint function $x + y = 8$ in light blue and the intersection function between them in black.



Graph 4.1: Utility function u(x,y)=3.16x$^{0.446}$y$^{0.554}$, constraint function x+y=8, their intersection function and its maxima plotted.

From Graph 4.1, we can tell that the coordinates which would give us the maximum utility under the constraint would be the coordinates of the intersection function's *maxima,* as shown by the blue point.

To find the answer coordinates, we would have to first find the intersection function by *expressing* our utility function in terms of one variable.

Making y the subject of our constraint function, we get

$$y = 8 - x$$

Substituting our constraint function into our utility function, we get

$$u(x, y) = 3.16x^{0.446}y^{0.554}$$

$$u(x, y) = 3.16x^{0.446}(8 - x)^{0.554}$$

Then we would have to find the intersection function's maxima by differentiating and optimising the above equation.

$$\frac{du}{dx} = 3.16(0.446)x^{-0.554}(8 - x)^{0.554} + 3.16x^{0.446}(0.554)(8 - x)^{-0.446}(-1)$$

$$\frac{du}{dx} = 1.41x^{-0.554}(8 - x)^{0.554} - 3.16x^{0.446}0.554(8 - x)^{-0.446}$$

$$1.41x^{-0.554}(8 - x)^{0.554} - 3.16x^{0.446}0.554(8 - x)^{-0.446} = 0$$

$$1.41x^{-0.554}(8 - x)^{0.554} = 3.16x^{0.446}0.554(8 - x)^{-0.446}$$

$$\frac{(8 - x)^{0.554}}{0.554(8 - x)^{-0.446}} = \frac{3.16x^{0.446}}{1.41x^{-0.554}}$$

$$\frac{8 - x}{0.554} = \frac{3.16x}{1.41}$$

$$8 - x = 1.24x$$

$$8 = 2.24x$$

$$\therefore \boldsymbol{x = 3.57}$$

$$\therefore y = 4.43$$

$$\therefore u = 12.7$$

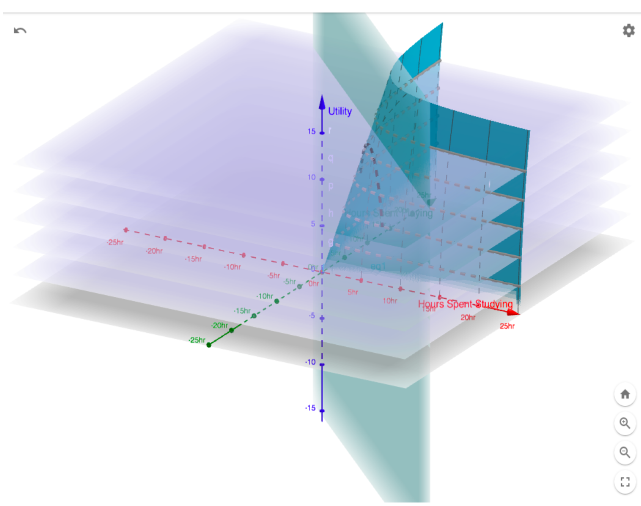
Therefore, (3.57,4.43,12.7) are our answer coordinates

3-Dimensional Analysis

However, differentiation was only possible because our constraint function was simple. In many instances, the intersection function obtained from the constraint function may be too complicated to differentiate easily. Hence, this problem could also be solved using *3D analysis*.

Manual

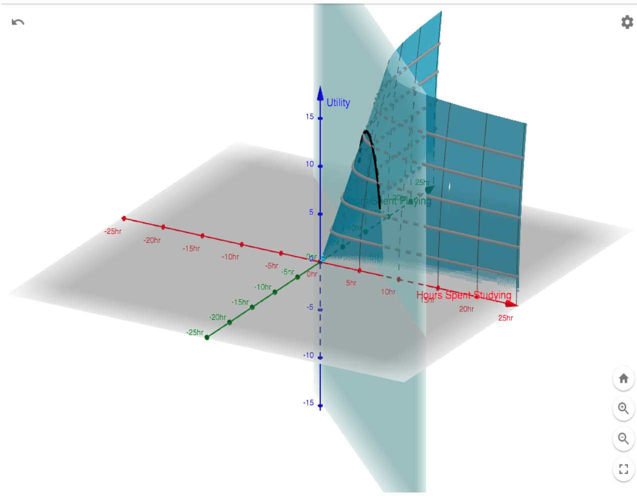
Let us look at Graph 4.1 again but this time, with contour lines. *Contour lines, also known as level curves, are lines on a two variable function whose x and y coordinates are such that when substituted into the function, they give a constant* $\Rightarrow u(x,y) = 3.16x^{0.446}y^{0.554} = k$. In other words, they are *cross-sections* of the function, made by *planes parallel* to the $x - y$ *plane*, as shown in Graphs 5.1 and 5.2.[10]



Graph 5.1: Utility function $u(x,y)=3.16x^{0.446}y^{0.554}$, constraint function x+y=8, intersection function,

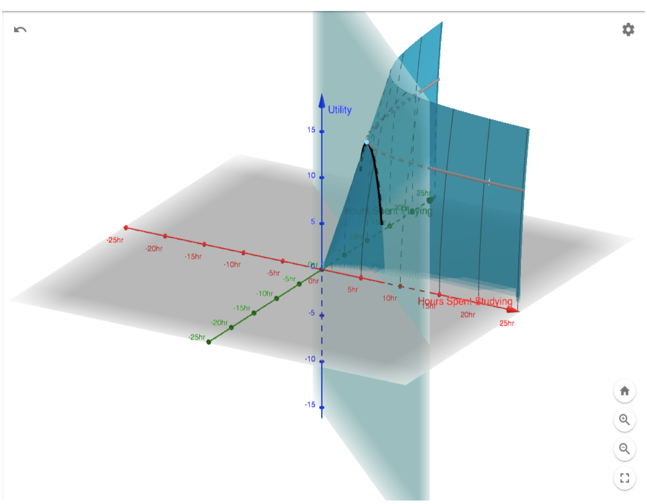
[10]"Level Sets." *Math Insight*, Math Insight, mathinsight.org/level_sets. Accessed 22 Oct. 2019.

planes parallel to the x-y plane and contour lines plotted.

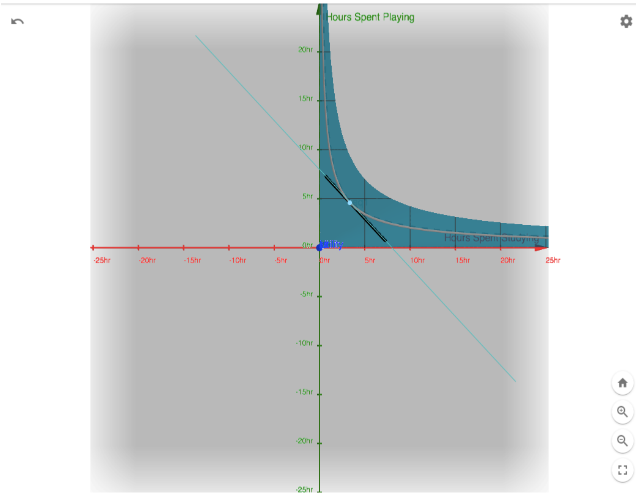


Graph 5.2: Utility function $u(x,y)=3.16x^{0.446}y^{0.554}$, constraint function $x+y=8$, intersection function and contour lines plotted.

We can tell that another way of finding the intersection function's maxima would be to find the contour line *tangential* to the constraint function and that, the *point of tangentiality* would provide us with the answer coordinates, as shown in Graphs 6.1 and 6.2.



Graph 6.1: Side view of utility function $u(x,y)=3.16x^{0.446}y^{0.554}$, constraint function $x+y=8$, intersection function, contour line tangential to the constraint function and point of tangentiality plotted.



Graph 6.2: Top view of utility function u(x,y)=3.16x$^{0.446}$y$^{0.554}$, constraint function x+y=8, intersection function, contour line tangential to the constraint function and point of tangentiality plotted.

To find the answer coordinates, we would have to express one variable in terms of the other for both, the function which represents the contour line and the constraint function.

$$u(x,y) = 3.16x^{0.446}y^{0.554} = k$$

$$y = \left(\frac{k}{3.16x^{0.446}}\right)^{\frac{1}{0.554}}$$

$$x + y = 8$$

$$y = 8 - x$$

Equating the two, we get an equation in k and x

$$y = \left(\frac{k}{3.16x^{0.446}}\right)^{\frac{1}{0.554}} = 8 - x$$

$$\frac{k^{1.81}}{8.02x^{0.807}} = 8 - x$$

$$k^{1.81} = 64.2x^{0.807} - 8.02x^{1.81} \rightarrow \textit{equation 1}$$

Equating the derivatives of the two functions, we get another equation in k and x

$$\frac{d\left(\frac{k^{1.81}x^{-0.807}}{8.02}\right)}{dx} = \frac{d(8-x)}{dx}$$

$$\frac{k^{1.81}-0.807x^{-1.81}}{8.02} = -1$$

$$k^{1.81}x^{-1.81} = 9.94 \rightarrow \textit{equation 2}$$

Solving using substitution, we get

$$(64.2x^{0.807} - 8.02x^{1.81})x^{-1.81} = 9.94$$

$$64.2x^{-1} - 8.02 = 9.94$$

$$\frac{1}{x} = 0.279$$

$$\therefore \boldsymbol{x = 3.57}$$

$$\therefore \boldsymbol{y = 4.43}$$

$$\therefore \boldsymbol{u = 12.7}$$

Therefore, (3.57,4.43,12.7) are our answer coordinates

Graphing Software

However, with *graphing software*, the 3D analysis can be made even simpler, as shown by the use of a *slider* in Figure 2.1. Graphing software also helps when one variable cannot be expressed in terms of the other to obtain the intersection.

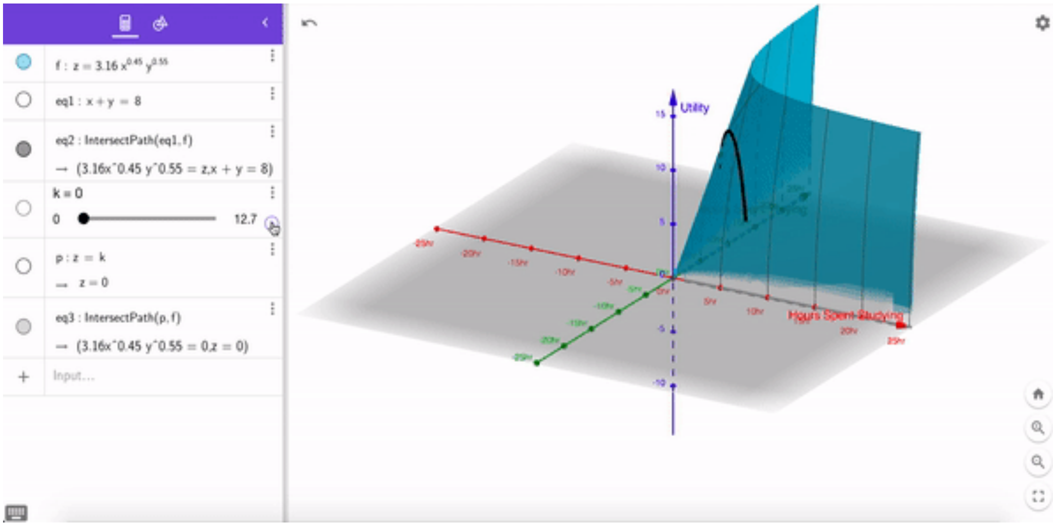


Figure 2.1: GIF of 3D graphing software view of contour line slider operation on plotted utility function u(x,y)= 3.16$x^{0.446}y^{0.554}$ and intersection function.

Link to view GIF: https://drive.google.com/file/d/1Wm5eScXz4ABR-UXaLXK_nIIHLuZ1hPLk/view?usp=sharing

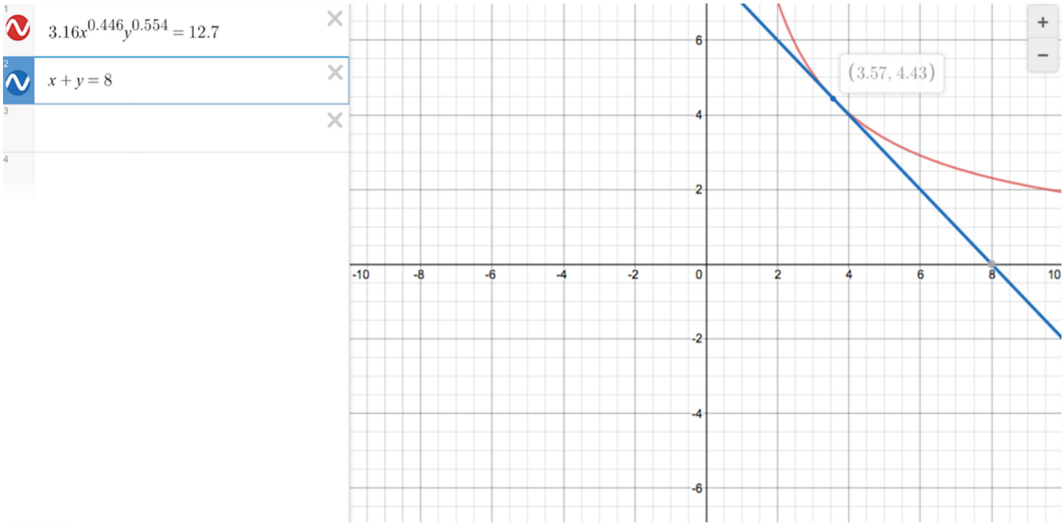
With the slider operation k can be found directly, without expressing it in terms of x. From the GIF, we see that when the slider of k reaches 12.7, the contour line becomes tangential to the intersection function. Thus, allowing us to find the function which represents the contour line.

$$u(x,y) = 3.16x^{0.446}y^{0.554} = k$$

$$k = 12.7$$

$$3.16x^{0.446}y^{0.554} = 12.7$$

Through plotting the function which represents the contour line and the constraint function $x + y = 8$, we can obtain the answer coordinates from the point of tangentiality, as shown in Graph 7.1.



Graph 7.1: Graphing software view of utility function u(x,y)=3.16x$^{0.446}$y$^{0.554}$ and constraint function x+y=8

$$\therefore \boldsymbol{x = 3.57}$$

$$\therefore \boldsymbol{y = 4.43}$$

$$\therefore \boldsymbol{u = 12.7}$$

Therefore, (3.57,4.43,12.7) are our answer coordinates

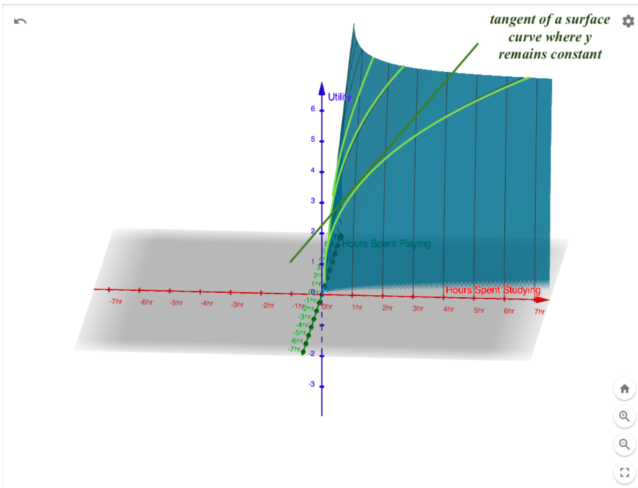
<u>The Lagrangian</u>

However, if neither of the variables can be made expressed in terms of the other and the problem must be solved manually, the *Lagrangian* must be used.

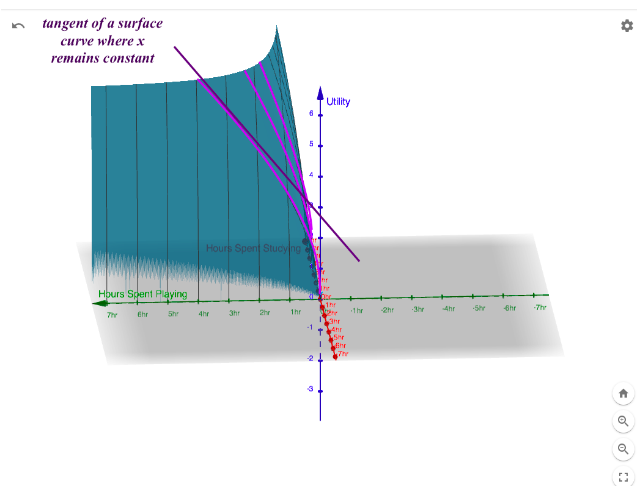
<u>*Manual*</u>

Let us look at Graph 6.1 and 6.2, once again and recall that the point of tangentiality between our utility function and constraint function provides us with the answer coordinates. To find the coordinates without the earlier two methods, we would have to borrow a few tools and techniques from vector calculus, namely, *partial derivatives, gradient vectors* and *the Lagrange multiplier.*

Partial derivative: For a function of two variables $u(x,y)$*, the partial derivative* $\frac{\partial\, u(x,y)}{\partial x}$ *measures* $u(x,y)'s$ *rate of change per unit change in x as y remains constant whereas the partial derivative* $\frac{\partial\, u(x,y)}{\partial y}$ *measures* $u(x,y)'s$ *rate of change per unit change in y as x remains constant.*[11] To illustrate this concept, I have included Graphs 8.1 and 8.2.



Graph 8.1: Tangent of utility function $u(x,y)=3.16x^{0.446}y^{0.554}$ ***at a certain point*** (x_1, y_1) ***with gradient*** $\frac{\partial\, u(x,y)}{\partial x}$ ***where y remains constant.***



[11] Hammond, Peter, et al. "Functions of Many Variables." *Essential Mathematics for Economic Analysis*, Fourth Edition, Pearson, 2012, p. 384.

Graph 8.2: Tangent of utility function $u(x,y)=3.16x^{0.446}y^{0.554}$ at a certain point (x_1, y_1) with gradient $\frac{\partial u(x,y)}{\partial y}$ where x remains constant.

*Gradient vector: For a function of two variables $u(x,y)$ with rates of change $\frac{\partial u(x,y)}{\partial x}$ and $\frac{\partial u(x,y)}{\partial y}$ in the **i** and **j** direction, the gradient vector*

$$\nabla u = \begin{bmatrix} \frac{\partial u(x,y)}{\partial x} \\ \frac{\partial u(x,y)}{\partial y} \end{bmatrix}$$

The gradient vector ∇u has many geometric properties, particularly that it is perpendicular to the contour line $u(x,y) = k$ and thus, points in the direction of steepest ascent.[12]

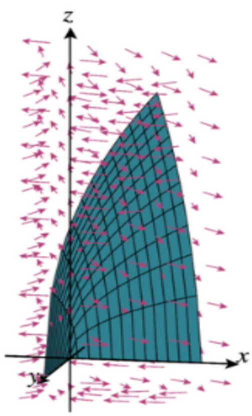
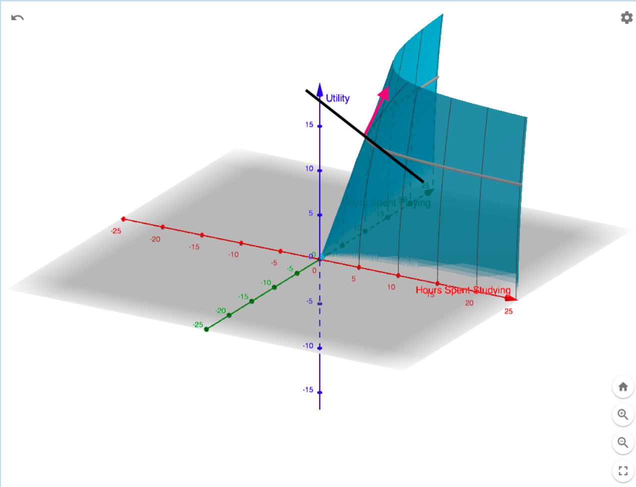


Figure 3.1: The Cobb Douglas function with its gradient vector field, not drawn to scale.

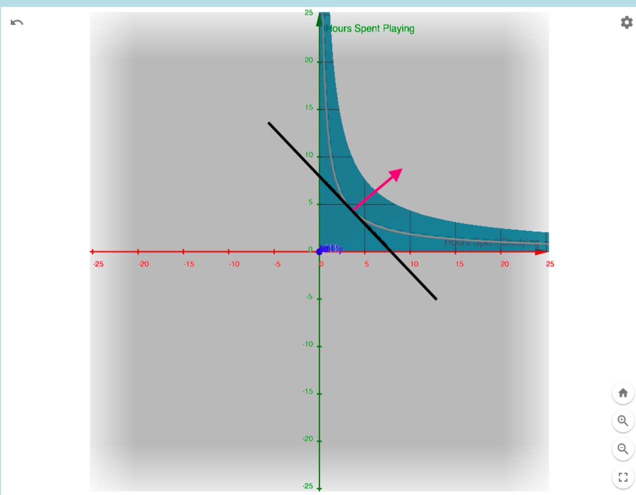
Proof: Gradient vectors are perpendicular to contour lines i.e. $\nabla u \perp u(x,y) = k$

[12] "Gradient: Definition and Properties." *MIT OpenCourseWare*, Massachusetts Institute of Technology, 2010, ocw.mit.edu/courses/mathematics/18-02sc-multivariable-calculus-fall-2010/2.-partial-derivatives/part-b-chain-rule-gradient-and-directional-derivatives/session-35-gradient-definition-perpendicular-to-level-curves/MIT18_02SC_notes_18.pdf. Accessed 30 Oct. 2020.

To prove this, it is sufficient to prove that gradient vectors are perpendicular to the tangents of contour lines, as shown in Graphs 9.1 and 9.2.



Graph 9.1: Side view of gradient vectors perpendicular to contour lines and their tangents.



Graph 9.2: Top view of gradient vectors perpendicular to contour lines and their tangents.

As shown in Graph 9.2, the gradient of the tangent of a contour line $u(x, y) = k$ *is given by* $\frac{dy}{dx}$.

$$\nabla u = \begin{bmatrix} \frac{\partial\, u(x,y)}{\partial x} \\ \frac{\partial\, u(x,y)}{\partial y} \end{bmatrix} \therefore \textit{ the gradient of } \nabla u \textit{ is } \frac{\frac{\partial\, u(x,y)}{\partial y}}{\frac{\partial\, u(x,y)}{\partial x}}.$$

Consider y to be a function of x such that y can be rewritten as $y(x)$ *and* $u(x, y)$ *can be rewritten as* $u(x, y(x))$.

To find $\frac{dy}{dx}$, $u(x, y(x)) = k$ *must be implicitly differentiated with respect to x using the multivariable chain rule.*

Single variable chain rule is

$$\frac{d\, u\big(y(x)\big)}{dx} = \frac{d\, u\big(y(x)\big)}{d\, y(x)} \bullet \frac{d\, y(x)}{dx}$$

By intuition, multivariable chain rule is[13]

$$\frac{d\, u\big(x, y(x)\big)}{dx} = \frac{\partial\, u\big(x, y(x)\big)}{\partial x} \bullet \frac{dx}{dx} + \frac{\partial\, u\big(x, y(x)\big)}{\partial\, y(x)} \bullet \frac{d\, y(x)}{dx}$$

$$u(x, y(x)) = k$$

$$\frac{d\, u\big(x,y(x)\big)}{dx} = 0$$

$$\frac{\partial\, u\big(x, y(x)\big)}{\partial x} \bullet \frac{dx}{dx} + \frac{\partial\, u\big(x, y(x)\big)}{\partial\, y(x)} \bullet \frac{d\, y(x)}{dx} = 0$$

$$\frac{\partial\, u\big(x, y(x)\big)}{\partial x} + \frac{\partial\, u\big(x, y(x)\big)}{\partial\, y(x)} \bullet \frac{d\, y(x)}{dx} = 0$$

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[13] Strang and Herman. "The Chain Rule for Multivariable Functions." Mathematics LibreTexts., OpenStax CNX, math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(OpenStax)/14%3A_Differentiation_of_Functions_of_Several_Variables/14.5%3A_The_Chain_Rule_for_Multivariable_Functions. Accessed 12 Dec. 2020.

$$\frac{dy(x)}{dx} = -\frac{\frac{\partial\, u(x,y(x))}{\partial x}}{\frac{\partial\, u(x,y(x))}{\partial\, y(x)}}$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial\, u(x,y)}{\partial x}}{\frac{\partial\, u(x,y)}{\partial\, y}}$$

The product of gradients perpendicular to each other is -1.

gradient of tangent • gradient of gradient vector

$$= -\frac{\frac{\partial\, u(x,y)}{\partial x}}{\frac{\partial\, u(x,y)}{\partial\, y}} \bullet \frac{\frac{\partial\, f(x,y)}{\partial y}}{\frac{\partial\, f(x,y)}{\partial\, x}}.$$

$$= -1$$

$\therefore QED$[14]

Table 2.1: Proof of gradient vectors being perpendicular to contour lines.

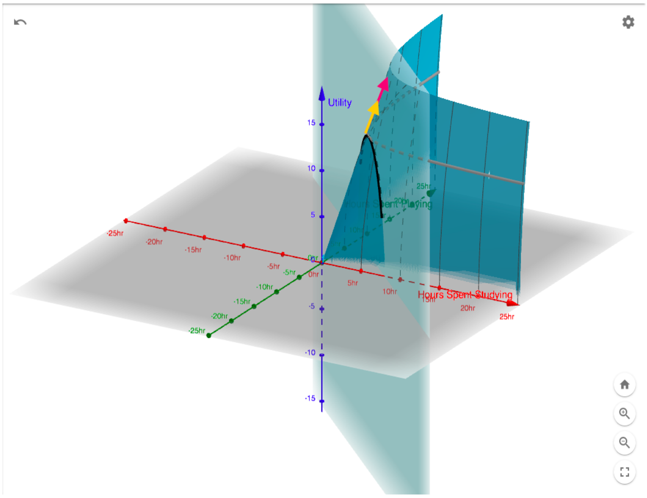
Note, because gradient vectors are *perpendicular* to their contour lines, gradient vectors of two contour lines tangential to each other would be *parallel* to each other at the point of tangentiality .

Lagrange multiplier: For function $u(x, y)$ *constrained by function* $g(x, y) = k$*, where their contour lines are tangential to each other, gradient vectors* $\boldsymbol{\nabla} u$ *and* $\boldsymbol{\nabla} g$ *are parallel to each other. Therefore, there exists a Lagrange multiplier* λ*, such that* $\boldsymbol{\nabla} u = \lambda \boldsymbol{\nabla} g$*, as shown in Graph 10.1 and 10.2*[15]

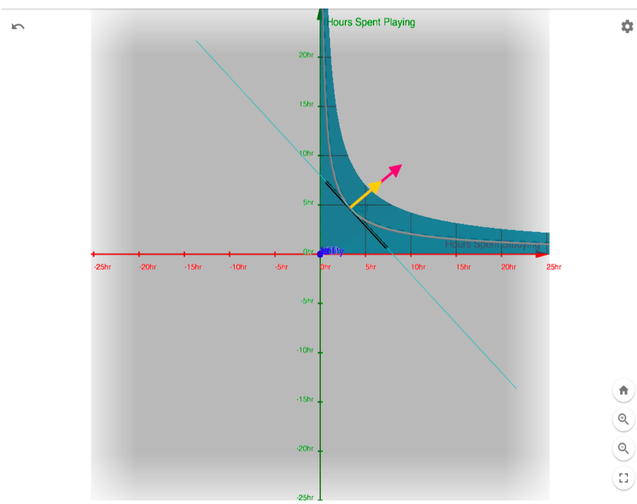
[14] "The Gradient and the Level Curve." People.Whitman.Edu, Whitman People, people.whitman.edu/~hundledr/courses/M225S09/GradOrth.pdf. Accessed 26 Nov. 2020.

[15] "Proof of Lagrange Multipliers." *MIT OpenCourseWare*, Massachusetts Institute of Technology, 2010, ocw.mit.edu/courses/mathematics/18-02sc-multivariable-calculus-fall-2010/2.-partial-derivatives/part-c-lagrange-multipliers-and-constrained-differentials/session-40-proof-of-lagrange-multipliers/MIT18_02SC_notes_22.pdf. Accessed 30 Oct. 2020.

Note, the constraint function is $g(x, y) = k$. In this case, $x + y = 8$ is the constraint function. Thus, $g(x, y) = k$ and $k = 8$.



Graph 10.1: Side view of parallel gradient vectors of utility function u(x,y) = 3.16x$^{0.446}$y$^{0.554}$ at point of tangentiality between utility function and constraint function.



Graph 10.2: Side view of parallel gradient vectors of utility function u(x,y) = 3.16x$^{0.446}$y$^{0.554}$ at point of tangentiality between utility function and constraint function.

As shown Graph 10.1 and 10.2, our utility function and constraint function have a point of tangentiality. Hence, $\nabla u = \lambda \nabla g$.[16] Hence, the answer coordinates can be found through equating the two gradient vectors.

$$u(x, y) = 3.16x^{0.446}y^{0.554}$$

$$g(x, y) = x + y$$

$$\nabla u = \lambda \nabla g$$

$$\begin{bmatrix} \frac{\partial\, u(x,y)}{\partial x} \\ \frac{\partial\, u(x,y)}{\partial \mathrm{y}} \end{bmatrix} = \begin{bmatrix} \frac{\lambda\, \partial\, g(x,y)}{\partial x} \\ \frac{\lambda\, \partial\, g(x,y)}{\partial \mathrm{y}} \end{bmatrix}$$

Equating the x component and y component of the gradient vectors and simplifying them, we get two equations

$$3.16(0.446)x^{-0.554}y^{0.554} = \lambda(1+0) \Rightarrow 3.16(0.446)x^{-0.579}y^{0.579} = \lambda \rightarrow \textit{equation 1}$$

$$3.16x^{0.446}(0.554)y^{-0.446} = \lambda(0+1) \Rightarrow 3.16x^{0.446}(0.554)y^{-0.446} = \lambda \rightarrow \textit{equation 2}$$

By equating the two, we can express one variable in terms of the other

$$3.16(0.446)x^{-0.579}y^{0.579} = 3.16x^{0.446}(0.554)y^{-0.446}$$

$$0.446x^{-0.579}y^{0.579} = x^{0.446}0.554y^{-0.446}$$

$$y = \frac{0.554}{0.446}x$$

$$y = 1.24x$$

Substituting the constraint function, the equation can be simplified to that of a single variable and we get

[16]Hammond, Peter, et al. "Constrained Optimisation." *Essential Mathematics for Economic Analysis*, Fourth Edition, Pearson, 2012, p. 497-513.

$$x + y = 8$$

$$y = 8 - x$$

$$8 - x = 1.24x$$

$$8 = 2.24x$$

$$\therefore \boldsymbol{x = 3.57}$$

$$\therefore \boldsymbol{y = 4.43}$$

$$\therefore \boldsymbol{u = 12.7}$$

Therefore, (3.57,4.43,12.7) are our answer coordinates

<u>Computer Manipulation Software</u>[17]

The above method of solution can also be achieved using the *Lagrangian function* $\mathcal{L}(x, y, \lambda)$. The Lagrangian function is defined as $\mathcal{L}(x, y, \lambda) = u(x, y) - \lambda(g(x, y) - k)$ such that $\boldsymbol{\nabla}\mathcal{L} = 0$. It is a very convenient *matrix optimisation program* which consists of the same steps as above, thus simplifying the constrained optimisation for any computer to solve as shown below.

$$\mathcal{L}(x, y, \lambda) = u(x, y) - \lambda(g(x, y) - k)$$

$$\boldsymbol{\nabla}\mathcal{L} = 0$$

$$\begin{bmatrix} \dfrac{\boldsymbol{\partial}\, \mathcal{L}(x, y, \lambda)}{\boldsymbol{\partial} x} \\ \dfrac{\boldsymbol{\partial}\, \mathcal{L}(x, y, \lambda)}{\boldsymbol{\partial} y} \\ \dfrac{\boldsymbol{\partial}\, \mathcal{L}(x, y, \lambda)}{\boldsymbol{\partial} \lambda} \end{bmatrix} = \begin{bmatrix} \dfrac{\boldsymbol{\partial}\, u(x, y)}{\boldsymbol{\partial} x} - \dfrac{\lambda \boldsymbol{\partial}\, g(x, y)}{\boldsymbol{\partial} x} - 0 \\ \dfrac{\boldsymbol{\partial}\, u(x, y)}{\boldsymbol{\partial} y} - \dfrac{\lambda \boldsymbol{\partial}\, g(x, y)}{\boldsymbol{\partial} y} - 0 \\ 0 - (g(x, y) - k) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

[17]Hammond, Peter, et al. "Constrained Optimisation." *Essential Mathematics for Economic Analysis*, Fourth Edition, Pearson, 2012, p. 497-513.

$$\begin{bmatrix} \frac{\partial\, u(x,y)}{\partial x} \\ \frac{\partial\, u(x,y)}{\partial \mathrm{y}} \\ g(x,y) \end{bmatrix} = \begin{bmatrix} \frac{\lambda\, \partial\, g(x,y)}{\partial x} \\ \frac{\lambda\, \partial\, g(x,y)}{\partial \mathrm{y}} \\ k \end{bmatrix}$$

Reaching the same set of equations as before

Analysis and Evaluation

Result

Thus, it can be concluded that IB students, dependent on studying and playing, are most satisfied at utility 12.7 after spending approximately 3 and a half hours (3.57 hours) studying and 4 and a half hours (4.43 hours) playing, after school and before sleeping.

Validity and Reliability

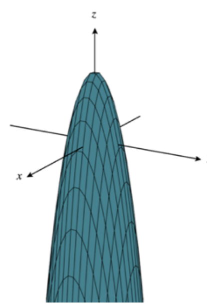
Primary and Secondary Sources

The main sources I used to conduct this investigation were "Intermediate Microeconomics: A Modern Approach" by Hal Varian and "Essential Mathematics for Economic Analysis" by Peter Hammond both of which are *reliable* sources written by *experts* in the field of Mathematical Economics, making them highly appropriate for my research. Hal Varian is currently the chief Economist of Google and Peter Hammond is an emeritus Professor of Economics at Stanford University. This lends both academics immense *authority* and *credibility*. All models, tools and findings borrowed from these books have been *validated* by reproving them in the body of this essay and have been *verified* by *cross-referencing* them with other resources like MIT OpenCourseWare and Harvard Scholar. Thus, adding *accuracy* to the results I obtained.

The Cobb Douglas Function

Despite the validity of my work, I realised that the Cobb Douglas function had a variety of limitations, which could have rendered my results unreliable. Particularly, the following three.

Limitations	*Improvements*
The first and most obvious limitation of the Cobb Douglas function is that it depends on only two variables.	One way of correcting this limitation would be to express the x and y as functions of other variables. For instance, if our happiness is dependent on time spent on interpersonal relationships (i), fulfilling personal goals (j), maintaining social media publicity (s) and making career progress (t), x can be expressed as $x(i, j)$ and y can be expressed as $y(s, t)$ where $x(i, j) = ci^a j^b$ and $y(s, t) = cs^a t^b$ while $u(x, y) = cx(i.j)^a y(s, t)^b$. However, while this increases the number of variables, it does so incrementally while still being limited by the Cobb Douglas function's form. Instead, it is possible to model this utility *stochastically* which would consider multiple variables and their *preference probabilities*. Note, while it remains possible to use functions of three variables or more to model utility, they become increasingly

	difficult to *visualise* and optimise in the *4th dimension*.
The second limitation is that the Cobb Douglas function has no *global maxima*. The form $u(x,y) = 3.16x^{0.446}y^{0.554}$ such that $\frac{\partial u}{\partial x} > 0$ and $\frac{\partial u}{\partial y} > 0$, hence the function is always *increasing*. Like our linear function, we know that it is unlikely our utility would increase without a satiation point.	To avoid this limitation, it is possible to model utility using functions which have global maxima, the most suitable of which would be the *paraboloid function* of the form $u(x,y) = \frac{-x^2}{a^2} + \frac{-y^2}{b^2} + c$  ***Figure 4.1: Paraboloid function plotted.*** However, it is possible that the data collected may not fit into such functions and hence, *data collection problems* may arise.
Last but not least, in many instances, the form of the Cobb Douglas function may be limiting. The random nature of the data collected on happiness could prevent data from agreeing with the assumptions or properties of the function, as proven by Amos Tversky	This limitation could be overcome best, by using stochastic utility models as they would be able to account for the randomness, multiplicity and extraneity of the data. Through using *probability distributions* or *probability spaces* with *measurable*

and other Economists. As mentioned earlier, there is also the issue of multiple and *extraneous variables*.	*random utility functions*, stochastic models can model preferences with the highest satisfaction accurately.[18] While this is still a developing field in *decision theory*, these models have made immense breakthroughs and continue to be an important stepping stone to optimising happiness.

Table 3.1: Limitations and improvements of the Cobb Douglas function.

However, it is important to acknowledge the Cobb Douglas function's high differentiability and manipulability as well as its simple Mathematics which makes it highly accessible for users. In fact, it was the reason it was chosen for this investigation.

Raw and Processed Data

Apart from the Cobb Douglas function, the data collected also had a few limitations of its own.

As I collected my data from a voluntary survey of 30 IB students, I realised the *small sample size* and inherent *self-selection bias* of this procedure could have made my findings *non-representative* of a larger group, limiting my reliability. Furthermore, the *correlation coefficient* (r) of my data was 0.617, as shown in Figure 1.1. This meant my relationship was only *moderately* strong.

These errors could have been mitigated through collecting more data points. However, the *self-selection bias* would have remained, as involuntary surveys violate *ethical guidelines*.

[18]Strzalecki, Tomasz. "Lectures on Stochastic Choice." Scholar.Harvard.Edu, Harvard University, 8 Mar. 2019, scholar.harvard.edu/files/tomasz/files/scslides34-handout.pdf. Accessed 15 Jan. 2021.

Methods of Mathematical Optimisation

The use of different methods to render the same results (3.57,4.43,12.7) consistently validated my solution and increased the accuracy of my work. Nonetheless, all three of the methods used had their own strengths and limitations in arriving at the solution.

Method of Solution	*Strengths*	*Limitations*
Differentiation	Differentiation is the most simple and accessible method.	However, it is greatly limited by requiring the simplification of the intersection function to that of a single variable. In the case that one variable cannot be expressed in terms of the other this method will not be applicable. Moreover, in the case that the constraint function is of a higher degree or more complicated, this method would be incredibly difficult to use.
3-Dimensional Analysis	3D analysis is relatively accessible as it uses the same concepts of differentiation. With the use of graphing software, it is even more	However, manually this method is limited by requiring simplification to that of a single variable.

	accessible as differentiation is not needed. It is more effective than differentiation as it can solve the problem even if the constraint function is complicated and not easily differentiable. With the use of graphing software, it can even solve the problem without expressing one variable in terms of the other.	Furthermore, even with graphing software, if the number of dependent variables increases, this method will not apply.
Lagrangian	By far, the Lagrangian is the strongest method. It does not require simplification to that of a single variable and can easily solve constrained optimisation problems when functions are complicated or have several variables.	However, this method is still highly limiting in that it can only optimise to find regional maxima and not global maxima.[19] Furthermore, it does not apply to functions which are non-differentiable. Note, this method is very tedious to do manually and even using computer

[19] J, Matt. "Discuss the Advantages and Disadvantages of the Method of Lagrange Multipliers." www.numerade.com, Numerade, www.numerade.com/questions/discuss-the-advantages-and-disadvantages-of-the-method-of-lagrange-multipliers-compared-with-solving. Accessed 27 Jan. 2021.

		manipulation software can be inaccessible.

Table 3.2: Strengths and limitations of the methods of solution.

Other Applications of the Cobb Douglas Function

The Cobb Douglas function is a well-established introductory model used in *Econometrics* to represent production relationships and preference rankings. It is conventionally used to study production relationships between labour and capital or consumer preferences between certain goods and services. It is widely employed by firms and countries for *constrained optimisation problems*, particularly because the exponents of the function have *economic significance*.[20]

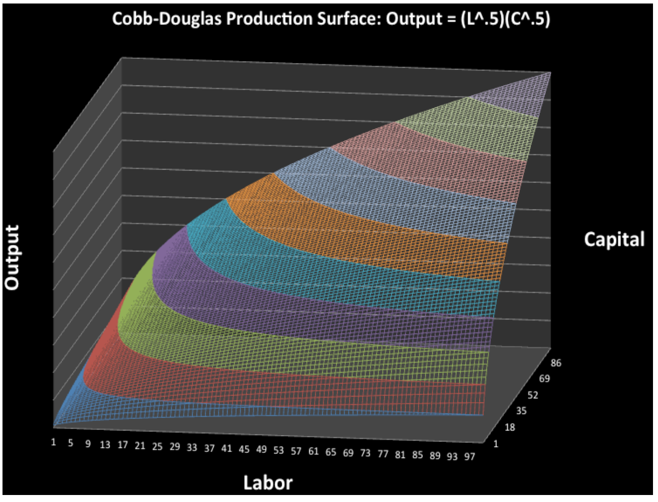


Figure 3.1: The Cobb Douglas function being used to optimise production.[21]

[20]"Utility" *Intermediate Microeconomics: A Modern Approach*, edited by Jack Repcheck, Eighth Edition, W.W Norton & Company, 2010, p. 54-69.

[21]"Cobb-Douglas Visualisation – I." *Stable Markets*, Wordpress, stablemarkets.wordpress.com/2013/06/11/cobb-douglas-visualisation-i. Accessed 11 June 2013. Accessed Feb 1. 2021

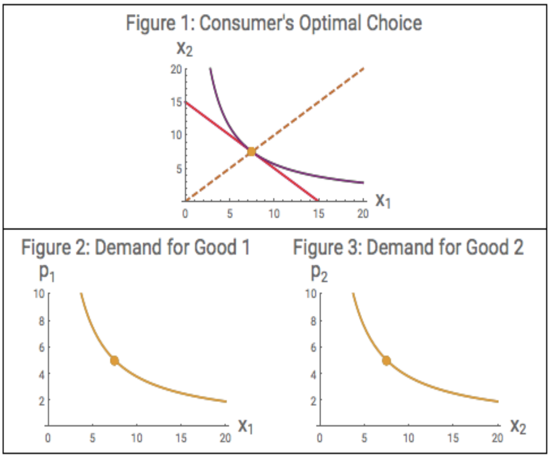


***Figure 3.2: The Cobb Douglas function being used to optimise consumer preferences.*[22]**

[22]"Cobb-Douglas Utility Function." *Wolfram Demonstrations Project*, Wolfram Demonstrations Project, demonstrations.wolfram.com/CobbDouglasUtilityFunction. Accessed 1 Feb. 2021.

Conclusion

To answer our research question, we can conclude that by using differentiation, 3-dimensional analysis and the Lagrangian we can most definitely optimise student satisfaction. While these methods and the Cobb Douglas function have their own shortcomings, it is worthy to recognise their mathematical properties which enable them to do such powerful real world computation in the first place.

This essay has successfully uncovered the means to student satisfaction and the errors which may limit it, thus demonstrating the power of Mathematics and the path it must take to improve.

Works Cited and Bibliography

Works Cited

- "Cobb-Douglas Utility Function." *Wolfram Demonstrations Project*, Wolfram Demonstrations Project, demonstrations.wolfram.com/CobbDouglasUtilityFunction. Accessed 1 Feb. 2021.
- "Cobb-Douglas Visualisation – I." *Stable Markets*, Wordpress, stablemarkets.wordpress.com/2013/06/11/cobb-douglas-visualisation-i. Accessed 11 June 2013. Accessed Feb 1. 2021
- "Gradient: Definition and Properties." *MIT OpenCourseWare*, Massachusetts Institute of Technology, 2010, ocw.mit.edu/courses/mathematics/18-02sc-multivariable-calculus-fall-2010/2.- partial-derivatives/part-b-chain-rule-gradient-and-directional-derivatives/session-35-gradient- definition-perpendicular-to-level-curves/MIT18_02SC_notes_18.pdf. Accessed 30 Oct. 2020.
- "Level Sets." *Math Insight*, Math Insight, mathinsight.org/level_sets. Accessed 22 Oct. 2019.
- "Preferences." *Intermediate Microeconomics: A Modern Approach*, edited by Jack Repcheck, Eighth Edition, W.W Norton & Company, 2010.
- "Proof of Lagrange Multipliers." *MIT OpenCourseWare*, Massachusetts Institute of Technology, 2010, ocw.mit.edu/courses/mathematics/18-02sc-multivariable-calculus-fall-2010/2.-partial- derivatives/part-c-lagrange-multipliers-and-constrained-differentials/session-40-proof-of-lagrange-multipliers/MIT18_02SC_notes_22.pdf. Accessed 30 Oct. 2020.

- "The Gradient and the Level Curve." People.Whitman.Edu, Whitman People, people.whitman.edu/~hundledr/courses/M225S09/GradOrth.pdf. Accessed 26 Nov. 2020.

- "Utility." *Intermediate Microeconomics: A Modern Approach*, edited by Jack Repcheck, Eighth Edition, W.W Norton & Company, 2010.

- Hammond, Peter, et al. "Constrained Optimisation." *Essential Mathematics for Economic Analysis*, Fourth Edition, Pearson, 2012, p. 497-513.

- Hammond, Peter, et al. "Functions of Many Variables." *Essential Mathematics for Economic Analysis*, Fourth Edition, Pearson, 2012, p. 384.

- J, Matt. "Discuss the Advantages and Disadvantages of the Method of Lagrange Multipliers." www.numerade.com, Numerade, www.numerade.com/questions/discuss-the-advantages-and- disadvantages-of-the-method-of-lagrange-multipliers-compared-with-solving. Accessed 27 Jan. 2021.

- Riveros, John. "A Brief Example to Model the Cobb-Douglas Utility Function Using Stata." *MSR Economic Perspective*, MSR Economic Perspective, 19 Dec. 2019, blog.ms- researchhub.com/2019/12/19/a-brief-example-to-model-the-cobb-douglas-utility-function-using-stata. Accessed 30 Sep. 2020.

- Strang and Herman. "The Chain Rule for Multivariable Functions." Mathematics LibreTexts., OpenStax CNX, math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(OpenStax)/14%3A_Differentiation_of_Functions_of_Several_Variables/14.5%3A_The_Chain_Rule_for_Multivariable_Functions. Accessed 12 Dec. 2020.

- Strzalecki, Tomasz. "Lectures on Stochastic Choice." Scholar.Harvard.Edu, Harvard University, 8 Mar. 2019, scholar.harvard.edu/files/tomasz/files/scslides34-handout.pdf. Accessed 15 Jan. 2021.

- Tragakes, Ellie. "Chapter 2: Competitive Markets: Demand and Supply." Economics for the IB Diploma, Second Edition, Cambridge University Press, 2012, p. 22.

Bibliography

- Khan, Salman. "Lagrange Multiplier Example, Part 1." *Khan Academy*, uploaded by Grant Sanderson, 11 Feb. 2017, www.khanacademy.org/math/multivariable-calculus/applications-of-multivariable-derivatives/lagrange-multipliers-and-constrained-optimization/v/lagrange-multiplier-example-part-1.

Appendix

Raw Data

Candidate Responses	x	y	$u\%$	u
A	3	11	80	19.2
B	4	2	60	14.4
C	4	3	35	8.40
D	6	4	65	15.6
E	2	6	38	9.12
F	5	3	40	9.60
G	2	4	85	20.4
H	9	2	45	10.8
I	7	10	85	20.4
J	3	6	80	19.2
K	7	10	85	20.4
L	7	10	85	20.4
M	6	4	65	15.6
N	3	8	80	19.2

O	4	6	85	20.4
P	3	7	85	20.4
Q	3	4	75	18.00
R	5	6	35	8.40
S	5	3	25	6.00
T	2	4	75	18.0
U	6	2	85	20.4
V	6	8	75	18.0
W	3	6	75	18.0
X	7	6	65	15.6
Y	6	9	85	20.4
Z	5	4	60	14.4
A1	3	11	50	12.0
B1	3	6	56	13.4
C1	5	6	40	9.60

D1	6	3	65	15.6

Table 1.1: Completed collected raw data on hours spent studying, playing and level of happiness

<u>Processed Data</u>

			Coordinates	
lnx	*lny*	*lnu*	*X = lnx-lny*	*Y = lnu-lny*
1.098612289	2.397895273	2.954910279	-1.299282984	0.5570150062
1.386294361	0.6931471806	2.667228207	0.6931471806	1.974081026
1.386294361	1.098612289	2.128231706	0.2876820725	1.029619417
1.791759469	1.386294361	2.747270914	0.4054651081	1.360976553
0.6931471806	1.791759469	2.210469804	-1.098612289	0.4187103349
1.609437912	1.098612289	2.261763098	0.5108256238	1.16315081
0.6931471806	1.386294361	3.015534901	-0.693147181	1.62924054
2.197224577	0.6931471806	2.379546134	1.504077397	1.686398954
1.945910149	2.302585093	3.015534901	-0.356674944	0.7129498079
1.098612289	1.791759469	2.954910279	-0.693147181	1.16315081
1.945910149	2.302585093	3.015534901	-0.356674944	0.7129498079
1.945910149	2.302585093	3.015534901	-0.356674944	0.7129498079
1.791759469	1.386294361	2.747270914	0.4054651081	1.360976553

1.098612289	2.079441542	2.954910279	-0.980829253	0.8754687374
1.386294361	1.791759469	3.015534901	-0.405465108	1.223775432
1.098612289	1.945910149	3.015534901	-0.847297860	1.069624752
1.098612289	1.386294361	2.890371758	-0.287682073	1.504077397
1.609437912	1.791759469	2.128231706	-0.182321557	0.3364722366
1.609437912	1.098612289	1.791759469	0.5108256238	0.6931471806
0.6931471806	1.386294361	2.890371758	-0.693147181	1.504077397
1.791759469	0.6931471806	3.015534901	1.098612289	2.32238772
1.791759469	2.079441542	2.890371758	-0.287682073	0.8109302162
1.098612289	1.791759469	2.890371758	-0.693147181	1.098612289
1.945910149	1.791759469	2.747270914	0.1541506798	0.955511445
1.791759469	2.197224577	3.015534901	-0.405465109	0.8183103235
1.609437912	1.386294361	2.667228207	0.2231435513	1.280933845
1.098612289	2.397895273	2.48490665	-1.299282984	0.0870113770
1.098612289	1.791759469	2.598235335	-0.693147181	0.8064758659
1.609437912	1.791759469	2.261763098	-0.182321557	0.4700036292
1.791759469	1.098612289	2.747270914	0.6931471806	1.648658626

Table 1.2: Completed processed data on natural logarithm of hours spent studying, playing and level of happiness along with coordinates X and Y.